Stable Limit Cycles due to Nonlinear Damping in Dual-Spin Spacecraft

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It is demonstrated that in the presence of energy dissipators on both rotor and despun platform of a dual-spin spacecraft it is possible for the vehicle to be unstable in its nominal attitude with rotor spin axis inertially stationary, but stable in a nearby state of free precession. Heuristic "energy sink" arguments are used to predict the indicated stability behavior for systems with various kinds of damping nonlinearities, and these predictions are confirmed by digital computer simulation. The similar anomalous flight behavior of Tacsat 1 is cited as evidence of the practical implications of this phenomenon.

Nomenclature§

a = distance from bearing axis to path of oscillating particle on B
B, B^* = rigid body and mass center of same
c = damping constant
D = damping force for oscillator on B, Eq. (21)
$\dot{E}$ = rate of change of mechanical energy in B
G = combination of gamma functions, Eq. (30)
I_3, I_1 = moments of inertia of B for bearing axis and transverse axis through B^*, respectively
k = oscillator spring constant on B
M, m = masses of B and particle oscillator on B
m_b = mass of each of three particles fixed to B, equaling the sum of m and the mass of associated oscillator housing
n = power identifying damping law, Eq. (21)
r = $(k/m)^{1/2}/|\lambda|$
X_1, X_2, X_3 = triad of orthogonal axes fixed in B, with X_3 the symmetry axis of B and the bearing axis common to B and B'
ρ = m/M_T
ζ = equivalent viscous damping ratio for oscillator on B
A = $I_1 + I_1' + 4m_b a^2 + 4m_b' a'^2 + (M' + 4m_b')(1 - \nu)l^2$
C = $I_3 + I_3' + 4m_b a^2 + 4m_b' a'^2$
J_3' = $I_3' + 4m_b' a'^2$
M_T = $M + M' + 4m_b + 4m_b'$
l = distance from B^* to $B^{*\prime}$
l_1, l_2 = $l(M + 4m_b)/M_T$ and $l'(M' + 4m_b')/M_T$, respectively
q, q' = C/A and J_3'/A, respectively
β = $a\omega_0(\Omega - \lambda)\lambda^{-2}[(r^2 - 1)^2 + (2\zeta r)^2]^{-1/2}$
β' = $a'\omega_0(\Omega + \sigma - \lambda')\lambda'^{-2}[(r'^2 - 1)^2 + (2\zeta' r')^2]^{-1/2}$
λ = $(q - 1)\Omega + q'\sigma$, "nutation rate" in body B
λ' = $(q - 1)\Omega + (q' - 1)\sigma$, "nutation rate" in body B'
Ω = nominal value of ω_3
σ = constant angular rate of B' relative to B
θ = nutation angle
ξ, ν, ψ = $\rho z + \rho' z'$, $(M' + 4m_b')/M_T$, and σt, respectively
$s\psi, c\psi$ = $\sin\psi$ and $\cos\psi$, respectively
$\omega_1, \omega_2, \omega_3$ = scalar components for X_1, X_2, X_3 of the inertial angular velocity of B

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§ Symbols a through ζ modified by primes represent like quantities associated with body B'.

Introduction

THE applicability of dual-spin attitude stabilization to prolate vehicles was indicated by an approximate and severely restricted analysis[1] in 1964, and in a less restricted oral presentation[2] in 1965. In 1967[3] this method gained nationwide recognition; and by 1969 many developers of spacecraft accepted it as a design alternative to be seriously considered, and it had been adopted for several flight vehicles. A number of research publications (e.g., Refs. 7–13) and volumes of internal company reports have appeared, but only recently have flight data for the first prolate dual-spin satellite (Tacsat 1) become publically available,[14] and it now appears that flight anomalies can be explained only in terms of system characteristics that have been omitted in all previous analyses.

For an electromechanical system as complex as a spacecraft, it is difficult to simplify the mathematical model of the system for analytical and computational convenience without sacrificing salient dynamic features. This paper illustrates the importance of accurately reflecting in analytical and experimental models the nature of the various energy dissipation mechanisms within the vehicle, including possible nonlinearities, which can lead to conclusions conceptually different from those indicated for linear systems. The flight performance of Tacsat 1 provides evidence that actual vehicles may behave in the manner indicated by the results to follow.

Understanding of the influence of energy dissipation on attitude stability has historically been greatly facilitated by heuristic arguments that have come to be known as "energy sink" methods of analysis. Briefly, the vehicle (whether a simple "spinner" or of dual-spin or multispin configuration) is initially assumed to consist of a minimum number of rigid elements incapable of energy dissipation, and the torque-free motion of the vehicle is determined. Next the presence of dissipative elements in the spacecraft is acknowledged, and the relative motions resulting in energy dissipation are calculated with the vehicle motion prescribed as that previously found appropriate when internal motions are suppressed. Finally, the resulting energy dissipation rates are used to reduce the kinetic energy of the vehicle as initially idealized. This iterative procedure is clearly not formally valid, and its conclusions are always subject to confirmation by more reliable methods, but it has proven invaluable in preliminary analysis.

Among the results in Ref. 4 is an energy-sink analysis yielding a general stability criterion obtained as an outgrowth of Iorillo's work.[2] If $\dot{E}$ is the rate of change of mechanical energy in the platform and $\dot{E}'$ is the corresponding rate in the rotor,¶ the stability requirement is

$$\dot{E}/\lambda + \dot{E}'/\lambda' < 0 \tag{1}$$

where λ and λ' are respectively the rates at which the angular velocity vector component transverse to the bearing axis rotates in the despun platform and the rotor. As will be shown, when the platform is sufficiently slow in its inertial rotation and the rotor spin-axis inertia is less than the vehicle transverse inertia, λ' is negative, while λ is positive. Since both $\dot{E}$ and $\dot{E}'$ are negative, one may then conclude that energy dissipation in the platform tends to stabilize the vehicle, while dissipation in the rotor is destabilizing. This much of the argument has been generally accepted for several years. It is the thesis of the present paper that in the presence of nonlinear damping laws the inequality (1) can be violated for motions in the immediate neighborhood of the nominal motion (producing instability of this motion), while for slightly larger coning angles (θ's) the same inequality can be satisfied (indicating cessation of increase in θ, or a form of stable limit cycle). This result depends only on the dependence of $\dot{E}$ and $\dot{E}'$ on the θ; if, for example, $\dot{E}$ increases quadratically with θ, while $\dot{E}'$ increases linearly, one is assured that the energy sink stability analysis will indicate the existence of a stable limit cycle.

The objectives of this paper can be met if, for a specific idealized model of a dual-spin vehicle, it can be shown explicitly that particular choices of damping laws indicate the existence of a stable state of precession in conjunction with an unstable nominal dual-spin motion, and if these results can be confirmed by digital simulation based on exact equations of motion.

Equations of Motion

We shall concentrate attention on a specific idealization of a dual-spin spacecraft (Fig. 1) previously appearing in the literature.[5] (The ensuing arguments apply as well conceptually to any other model which permits energy dissipation on both rotor and despun platform.) This model consists of two rigid bodies B and B' to which mass particles are attached, with one particle on each body free to oscillate on a linear spring on a track parallel to the unit vector $\mathbf{X}_3$ and to the bearing axis common to B and B'. The angular velocity of B' relative to B is given by $\sigma\mathbf{X}_3$, where σ is constrained to remain constant. Coordinates z and z', respectively, identify the excursions of the two oscillating particles (of masses m and m', and spring constants k and k') on bodies B and B'; when $z = z' = 0$ the vehicle is fully axisymmetric. In Ref. 5, these oscillations are resisted by viscous drag forces $-c\dot{z}\mathbf{X}_3$ and $-c'\dot{z}'\mathbf{X}_3$, respectively. Because these damping laws are linear, it was possible[5] to employ linearized equations of motion for stability analysis. The equations have periodic coefficients, and a numerical implementation of Floquet theory is employed[5] to obtain stability information. The exact nonlinear equations of motion of the system of Fig. 1, with linear damping,[5] provide the starting point for the present analysis. Modified only by replacing $-c\dot{z}\mathbf{X}_3$ and $-c'\dot{z}'\mathbf{X}_3$ by the general damping forces $D\mathbf{X}_3$ and $D'\mathbf{X}_3$, respectively, so that investigations can be made for a variety of forms of D and D', they are as follows (see Nomenclature):

$$\begin{aligned} &A\dot{\omega}_1 - (A - C)\omega_2\omega_3 + J_3'\sigma\omega_2 + 2M_T\xi\dot{\xi}\omega_1 + \\ &\quad M_T\xi^2(\dot{\omega}_1 - \omega_2\omega_3) + m\{-2(\xi + l_2) \times \\ &\quad [z(\dot{\omega}_1 - \omega_2\omega_3) + \dot{z}\omega_1] + z[-2\dot{\xi}\omega_1 + 2\dot{z}\omega_1 + \\ &\quad z(\dot{\omega}_1 - \omega_2\omega_3) - a(\dot{\omega}_3 + \omega_1\omega_2)]\} + \\ &\quad m'[-2(\xi - l_1)[z'(\dot{\omega}_1 - \omega_2\omega_3) + \dot{z}'\omega_1] + \\ &\quad z'[-2\dot{\xi}\omega_1 + 2\dot{z}'\omega_1 + z'(\dot{\omega}_1 - \omega_2\omega_3) - \\ &\quad a'\,\mathrm{c}\psi(\dot{\omega}_3 + \omega_1\omega_2)] + a'\,\mathrm{s}\psi\{\ddot{z}' + \\ &\quad z'[(\omega_3 + \sigma)^2 - \omega_2^2]\}] = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &A\dot{\omega}_2 - (C - A)\omega_1\omega_3 - J_3'\sigma\omega_1 + 2M_T\xi\dot{\xi}\omega_2 + \\ &\quad M_T\xi^2(\dot{\omega}_2 + \omega_1\omega_3) + m\{-2(\xi + l_2) \times \\ &\quad [z(\dot{\omega}_2 + \omega_1\omega_3) + \dot{z}\omega_2] + z[-2\dot{\xi}\omega_2 + 2\dot{z}\omega_2 + \\ &\quad z(\dot{\omega}_2 + \omega_1\omega_3)] - a[\ddot{z} + z(\omega_3^2 - \omega_1^2)]\} + \\ &\quad m'[-2(\xi - l_1)[z'(\dot{\omega}_2 + \omega_1\omega_3) + \dot{z}'\omega_2] + \\ &\quad z'[-2\xi\omega_2 + 2\dot{z}'\omega_2 + z'(\dot{\omega}_2 + \omega_1\omega_3)] - \\ &\quad a'\,\mathrm{c}\psi\{\ddot{z}' + z'[(\omega_3 + \sigma)^2 - \omega_1^2]\} - \\ &\quad a'\,\mathrm{s}\psi z'(\dot{\omega}_3 - \omega_1\omega_2)] = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} &C\dot{\omega}_3 - ma[2\dot{z}\omega_1 + z(\dot{\omega}_1 - \omega_2\omega_3)] - m'a' \times \\ &\quad \mathrm{s}\psi[2\dot{z}'\omega_2 + z'(\dot{\omega}_2 + \omega_1\omega_3)] - \\ &\quad m'a'\,\mathrm{c}\psi[2\dot{z}'\omega_1 + z'(\dot{\omega}_1 - \omega_2\omega_3)] = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} &m(1 - \rho)\ddot{z} - m'\rho\ddot{z}' - ma(\dot{\omega}_2 - \omega_1\omega_3) - \\ &\quad m(\omega_1^2 + \omega_2^2)[z(1 - \rho) - l_2 - \rho'z'] - \\ &\quad D + kz = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &-m\rho'\ddot{z} + m'(1 - \rho')\ddot{z}' + m'a'\{\mathrm{s}\psi[\dot{\omega}_1 + \omega_2(\omega_3 + \\ &\quad 2\sigma)] - \mathrm{c}\psi[\dot{\omega}_2 - \omega_1(\omega_3 + 2\sigma)]\} - \\ &\quad m'(\omega_1^2 + \omega_2^2)[z'(1 - \rho') + l_1 - \rho z] - \\ &\quad D' + k'z' = 0 \end{aligned} \tag{6}$$

Energy Sink Analysis

In the energy-sink approach, one begins by solving the vehicle equations of motion with internal moving parts con-

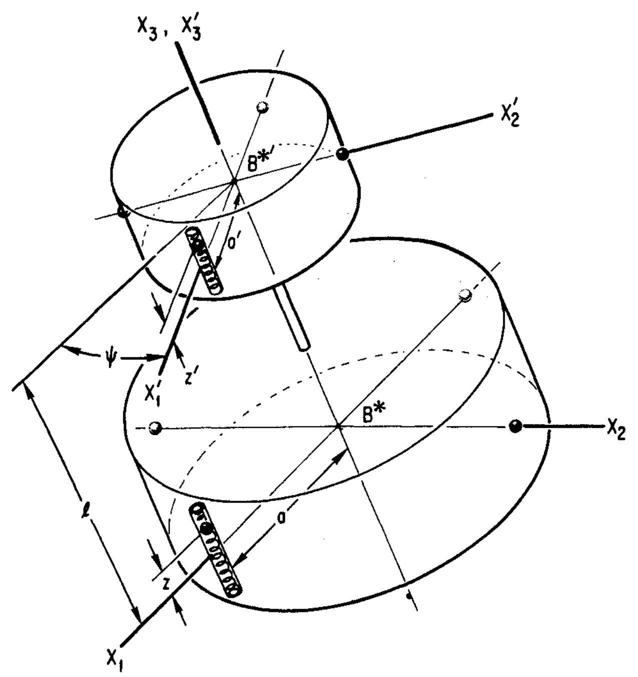

Fig. 1 Idealized dual-spin spacecraft.

¶ These heuristic concepts must be interpreted carefully. An energy dissipator or energy source on the platform (or rotor) contributes to $\dot{E}$ (or $\dot{E}'$) only insofar as it does work by virtue of the existence of a deviation from a perfect spinning motion. Torques paralleling the bearing axis (e.g., the motor torque implied by the constraint) do not contribute to $\dot{E}$ and $\dot{E}'$.

strained against relative motions; thus the coordinates z and z' in Eqs. (2–4) are set equal to zero. Equation (4) then says $\dot{\omega}_3 = 0$, and if ω_3 is replaced in Eqs. (2) and (3) by the constant Ω, these equations may be written

$$\dot{\omega}_1 + \lambda\omega_2 = 0; \ \dot{\omega}_2 - \lambda\omega_1 = 0 \qquad (7)$$

where with $q \equiv C/A$ and $q' \equiv J_3'/A$

$$\lambda \equiv [(C - A)\Omega + J_3'\sigma]/A = (q - 1)\Omega + q'\sigma \qquad (8)$$

Equations (7) have the solution

$$\omega_2 = \omega_0 \cos\lambda t; \ \omega_1 = -\omega_0 \sin\lambda t \qquad (9)$$

for initial conditions $\omega_2(0) = \omega_0$ and $\omega_1(0) = 0$. From this solution it is evident that λ may be interpreted as the rate at which the angular velocity component transverse to the bearing axis rotates in body B. This quantity is called by some writers the "nutation rate as measured in B." One could, of course, equally well write Eqs. (7) with B' playing the role adopted by B, by placing a prime on each symbol, with proper interpretation. If then λ' (the "nutation rate as measured in B'") were expressed in terms of the system parameters appearing in Eq. (8), the result would be

$$\lambda' = (q - 1)\Omega + (q' - 1)\sigma \qquad (10)$$

It now becomes apparent that, for sufficiently small angular rate Ω of body B, we always have $\lambda > 0$, and for $J_3' < A$, we have $\lambda' < 0$. Since, for the system presently assumed, $\dot{E}$ and $\dot{E}'$ are negative, criterion (1) indicates that for the assumed conditions the dissipation of energy in B is stabilizing while dissipation in B' is destabilizing. If now, in the spirit of the energy-sink method, it is assumed that there are damped linear oscillators on both B and B' and the solutions in Eqs. (7) are accepted as representative of the motion of the vehicle, one may solve the equations of motion of the oscillators (with prescribed base motion) and calculate $\dot{E}$ and $\dot{E}'$.

Linearization of Eq. (4) in the variables ω_1, ω_2, z, and z' indicates the constancy of $\omega_3 = \Omega$, and the relative rate σ is assumed constant from the outset as in the reference. The remaining equations of motion in the variables ω_1, ω_2, z, and z' become upon linearization in the variational coordinates

$$A\dot{\omega}_1 + [(C - A)\Omega + J_3'\sigma]\omega_2 + m'a' \sin\sigma t[\ddot{z}' + (\Omega + \sigma)^2 z'] = 0 \qquad (11)$$

$$A\dot{\omega}_2 - [(C - A)\Omega + J_3'\sigma]\omega_1 - ma(\ddot{z} + \Omega^2 z) - m'a' \cos\sigma t[\ddot{z}' + (\Omega + \sigma)^2 z'] = 0 \qquad (12)$$

$$m(1 - \rho)\ddot{z} - m\rho'\ddot{z}' - ma(\dot{\omega}_2 - \Omega\omega_1) - D + kz = 0 \qquad (13)$$

$$m'(1 - \rho')\ddot{z}' - m\rho'\ddot{z} + m'a'\{\sin\sigma t[\dot{\omega}_1 + (\Omega + 2\sigma)\omega_2] - \cos\sigma t[\dot{\omega}_2 - (\Omega + 2\sigma)\omega_1]\} - D' + k'z' = 0 \qquad (14)$$

Ignoring the terms involving z and z' in Eqs. (11) and (12), we have obtained solutions (9) for $\omega_1(t)$ and $\omega_2(t)$. It remains only to substitute these solutions into Eqs. (13) and (14) and to solve the resulting equations for z and z'. This step is facilitated by ignoring the small quantities ρ and ρ', thereby uncoupling the equations, and by assuming "equivalent viscous damping forces" for D and D'. The last assumption permits an approximate solution of Eqs. (13) and (14) to be obtained without restriction to any particular damping law. Any critical nonlinearity in a damper is preserved by representing the "equivalent viscous damping constant" as a function of oscillations amplitude. The resulting version of Eq. (13) becomes simply

$$\ddot{z} + 2\zeta p\dot{z} + p^2 z = a\omega_0(\Omega - \lambda) \sin\lambda t \qquad (15)$$

with $p^2 \equiv k/m$, and with the "equivalent viscous damping ratio" ζ a constant to be defined so as to provide approximately the same energy dissipation rate for a given excitation as would result from the specified damping force D. This concept will later be more precisely defined. Equation (15) has the steady-state solution

$$z = \beta \sin(\lambda t + \phi) \qquad (16)$$

where $\beta \equiv a\omega_0(\Omega - \lambda)\lambda^{-2}[(r^2 - 1)^2 + (2\zeta r)^2]^{-1/2}$; $r \equiv p/|\lambda|$, and $\phi \equiv \tan^{-1}[2\zeta r(1 - r)^{-1} \operatorname{sgn}(\lambda)]$. Similar calculations for the oscillator on B' provide

$$\ddot{z}' + 2\zeta' p'\dot{z}' + p'^2 z' = a'\omega_0(\Omega + \sigma - \lambda') \sin\lambda' t \qquad (17)$$

with $p'^2 \equiv k'/m'$ and a corresponding definition of ζ'. Equation (17) has the solution

$$z' = \beta' \sin(\lambda' t + \phi') \qquad (18)$$

where $\beta' \equiv a'\omega_0(\Omega + \sigma - \lambda')\lambda'^{-2}[(r'^2 - 1)^2 + (2\zeta' r')^2]^{-1/2}$, with r' and ϕ' defined in parallel with r and ϕ.

There remains the task of calculating the work done by the damping forces D and D' as the oscillator masses respond to the vehicle motion in the manner approximated by Eqs. (16) and (18). Attention is first restricted to a class of damping forces described by a power law, i.e.,

$$\begin{aligned} D &= -ca\eta|\dot{z}a^{-1}\eta^{-1}|^n \operatorname{sgn}(\dot{z}) \\ D' &= -c'a'\eta|\dot{z}'a'^{-1}\eta^{-1}|^{n'} \operatorname{sgn}(\dot{z}') \end{aligned} \qquad (19)$$

where sgn is the signum function, and $\eta \equiv |q\Omega + q'\sigma|$. Thus, a viscous force is represented by selecting the desired power (n or n') equal to one, a quadratic damping force corresponds to n(or n') $= 2$, and a Coulomb force for "dry friction" corresponds to the limiting case with n(or n') $= 0$.

In calculating work done by internal forces in bodies B and B', one must recognize that equal and opposite interaction forces between each particle and its housing do work; e.g., in body B, the work integral is

$$W(t) = \int_0^t [(\mathbf{F}^p \cdot \mathbf{V}^p) + (-\mathbf{F}^p \cdot \mathbf{V}^{p*})]d\tau \qquad (20)$$

where $\mathbf{F}^p$ is the force on the oscillating particle and $\mathbf{V}^p$ is the inertial velocity of that particle, and the reaction force $-\mathbf{F}^p$ is dot-multiplied by the velocity of the point of the housing coincident with the particle at a given time. The relationship

$$\mathbf{V}^p = \mathbf{V}^{p*} + \dot{z}\mathbf{X}_3 \qquad (21)$$

where $\mathbf{X}_3$ is a unit vector paralleling the bearing axis, permits the work integral in Eq. (20) to reduce to

$$W(t) = \int_0^t (-kz + D)\dot{z}d\tau \qquad (22)$$

where D may be substituted from Eq. (19). A similar expression can, of course, be written for $W'(t)$, the work done by the oscillator in B'.

The energy dissipation rates may be estimated by calculating the energy dissipated in a free precession (or nutation) cycle and then dividing by the corresponding period. This calculation provides (for any starting point τ_0)

$$\dot{E} = |\lambda| \ (2\pi)^{-1} \int_{\tau_0}^{\tau_0 + \tau_1} (-kz + D)\dot{z}d\tau \qquad (23)$$

where $\tau_1 \equiv 2\pi/|\lambda|$, and a corresponding equation in primed quantities for $\dot{E}'$.

The spring forces $-kz$ and $-k'z'$ contribute nothing to $\dot{E}$ and $\dot{E}'$, due to the conservative nature of the forces, as reflected in the orthogonality of the trigonometric functions. Substitution of Eq. (19) into Eq. (23) thus provides the equivalent integral (over one-quarter of the period)

$$\dot{E} = -2\pi^{-1}ca^2\eta^2 \int_0^{\pi/2} |(\dot{z}a^{-1}\eta^{-1})^{n+1}| d\varphi \qquad (24)$$

where $\varphi \equiv |\lambda|(\tau - \tau_0)$. Substitution of Eq. (16) into Eq. (24) now yields

$$\dot{E} = -2\pi^{-1}\eta^2 ca^2 G|\lambda\beta(a\eta)^{-1}|^{n+1} \qquad (25)$$

where τ_0 has been chosen as $\phi/|\lambda|$, and, in terms of gamma functions $\Gamma[\ ]$,

$$G \equiv \int_0^{\pi/2} \cos^{n+1}\varphi d\varphi = \frac{1}{2}\pi^{1/2}\frac{\Gamma[(n+2)/2]}{\Gamma[(n+3)/2]} \quad (26)$$

An expression for $\dot{E}'$ analogous to Eq. (25) may be derived using Eqs. (18) and (19). In terms of these expressions for $\dot{E}$ and $\dot{E}'$, the energy sink stability criterion, inequality (1) may be written directly, but a more useful form of this equation may be obtained by expressing the transverse angular velocity magnitude ω_0 in terms of θ:

$$\omega_0 = |(q\Omega + q'\sigma)| \tan\theta = \eta \tan\theta \quad (27)$$

With the substitution of Eqs. (16, 18, and 25) and its counterpart for $\dot{E}'$, and Eq. (27), inequality (1) becomes

$$\frac{ca^2G}{[(q-1)\Omega + q'\sigma]} \left|\frac{[(2-q)\Omega - q'\sigma]\tan\theta}{[(q-1)\Omega + q'\sigma][(r^2-1)^2 + (2\zeta r)^2]^{1/2}}\right|^{n+1} + \frac{c'a'^2G'}{[(q-1)\Omega + (q'-1)\sigma]} \times \left|\frac{[(2-q)\Omega + (2-q')\sigma]\tan\theta}{[(q-1)\Omega + (q'-1)\sigma][(r'^2-1)^2 + (2\zeta' r')^2]^{1/2}}\right|^{n'+1} > 0 \quad (28)$$

In past applications, attention has always focused on the case $n = n' = 1$, corresponding to viscous damping on both rotor and platform. In this special, wholly viscous case ζ and ζ' are simply constants; specifically these damping ratios are

$$\zeta = c/[2(km)^{1/2}] = c/2pm; \qquad \zeta' = c'/2p'm' \quad (29)$$

In generalizing the heuristic stability criterion (28) to accommodate nonlinear damping, one must recognize that the concept of "equivalent viscous damping" has validity only for a particular forcing function, as represented by the right hand sides of Eqs. (15) and (17). For oscillators with nonlinear damping on a given vehicle, the equivalent viscous damping ratios ζ and ζ' are not constant, since they depend on ω_0 (or θ). It is therefore required that the functional dependence of ζ and ζ' on system parameters and θ be established and the results substituted into Eq. (28) before this stability criterion is put to use. We define ζ and ζ' as those values which would result in the same rate of dissipation of energy in equivalent viscous oscillators under the same excitation. An expression for ζ will be developed, permitting a parallel expression for ζ' to be written by inspection.

Using Eq. (25) twice, one can obtain the two dissipation rates which are to be equated. Equating $\dot{E}$ from Eq. (25) to the special case of $\dot{E}$ when $n = 1$ yields (canceling common terms, and noting that $c = 2p\zeta m$ and $G = \pi/4$ for $n = 1$)

$$\tfrac{1}{4}\pi p\zeta m\beta^2\lambda^2(a\eta)^{-2} = cG|\lambda\beta(a\eta)^{-1}|^{n+1} \quad (30)$$

Expanding Eq. (30) and substituting for ω_0 from Eq. (27) provides

$$\{\lambda^2[(r^2-1)^2 + 4\zeta^2r^2]\}^{\frac{n-1}{2}} = 2cG(\pi p\zeta m)^{-1}|\tan\theta(\Omega - \lambda)|^{n-1} \quad (31)$$

Each side of Eq. (31) can be raised to the power $2/(n-1)$ and divided by λ^2 to obtain

$$\zeta^{2n/(n-1)}[4r^2(\pi pm)^{2/(n-1)}] + \zeta^{2/(n-1)}[(r^2-1)^2(\pi pm)^{2/(n-1)}] = (2cG)^{2/(n-1)}(\tan^2\theta)(\Omega-\lambda)^2/\lambda^2 \quad (32)$$

This result is invalid when $n = 1$, in which case Eq. (29) applies. By repeating the preceding derivation, one can obtain an equation for ζ' identical to Eq. (32) except that the factor $(\Omega - \lambda)^2$ is replaced by $(\Omega + \sigma - \lambda)^2$, and all parameters other than θ are primed.

Substitution of the solutions of these equations for ζ and ζ' into Eq. (28) provides the final energy-sink stability criterion. The aspect of this stability criterion most worthy of notice in the present context is the dependence of the result on θ. For the case $n' < n$, it is possible for this condition to be violated for θ near zero, but satisfied for θ above some finite value. The result might be termed a stable limit cycle at that value of θ for which the quantity in Eq. (28) is exactly zero.

In many applications of interest, one of the bodies (say B) is nominally inertially despun (so $\Omega = 0$), and damping ratios ζ and ζ' are so small as to have negligible influence in Eq. (28) as long as r and r' differ appreciably from unity. The stability criterion then becomes (assuming $\sigma > 0$, and canceling σ where possible)

$$\frac{ca^2G}{q'}\left|\frac{\tan\theta}{r^2-1}\right|^{n+1} + \frac{c'a'^2G'}{(q'-1)}\left|\frac{(2-q')\tan\theta}{(1-q')(r'^2-1)}\right|^{n'+1} > 0 \quad (33)$$

As an illustration of the significance of this result, consider the stability criterion for a vehicle with the inertia ratio $q' = \frac{1}{3}$, and with tuning ratios $r = r' = 2$. The criterion then becomes (canceling a factor of 3)

$$ca^2G|(\tfrac{1}{3})\tan\theta|^{n+1} - \tfrac{1}{2}c'a'^2G'|(\tfrac{5}{6})\tan\theta|^{n'+1} > 0 \quad (34)$$

For the special case $n = 1$ (viscous damping on the despun body B) and $n' \to 0$ (with $n' = 0$ representing the limiting case of Coulomb damping on the rotating body B') the functions G and G' are given by

$$G = \pi/4; \quad \lim_{n'\to 0} G' = 1 \quad (35)$$

The stability criterion is in the limit as $n' \to 0$ (canceling the factor $\frac{1}{12}$)

$$\tan\theta[\tan\theta - 15c'a'^2(\pi ca^2)^{-1}] > 0 \quad (36)$$

For convenience in subsequent discussion, the terms sublinear, linear, and superlinear are applied to damping forces proportional to velocity to powers <1, 1, and >1, respectively. It is evident from Eq. (36) that the system with $n = 1$ and $n' \to 0$ is unstable in the null solution, regardless of the values of c' and c, i.e., *no linear* (or viscous) damper on the despun body B can overcome the smallest *sublinear* damper on the rotating body B'. Note that Coulomb damping represents a limiting case of sublinear damping, but it presents special problems due to the stiction phenomenon, which is excluded from Eq. (36).

To obtain for the given system a limit cycle at $\theta = 1°$, we require from Eq. (36)

$$c'a'^2(ca^2)^{-1} = (\pi/15)\tan 1° = 0.00366 \quad (37)$$

This number may correspond to forces sufficiently small to tempt a casual investigator into the error of ignoring the damping on B', and yet it destroys the stability of the solution for $\theta = 0$, and introduces a variable-direction 1° error in the spin-axis orientation in inertial space.

In a practical application one must expect a certain amount of stiction in a Coulomb oscillator. This would mean that under sufficiently small excitation the Coulomb oscillator would not move at all, with no resulting energy dissipation. In Figs. 2–4 schematic portrayals of system behavior are shown in order to illustrate qualitatively the influence of Coulomb damping with stiction on the rotor, in combination with viscous damping on the platform.

Figure 2 shows three trajectories in the ω_1,ω_2 space, corresponding to three sets of initial conditions, represented by points A, B, and C. For sufficiently small perturbation (see

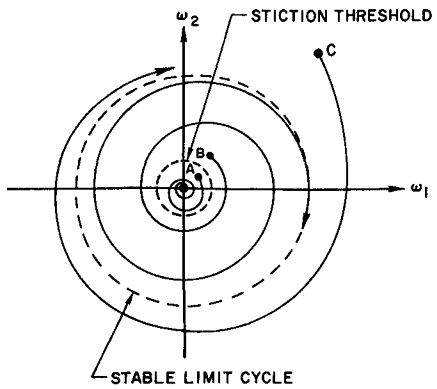

Fig. 2 Schematic of possible trajectories in transverse angular velocity space.

point A), the stiction threshold is not exceeded, and the nominal motion (represented by the origin) is asymptotically stable (but not globally so). Somewhat larger perturbation (see point B) initiates a growing departure from nominal motion, as $|\dot{E}'/\lambda'|$ exceeds $|\dot{E}/\lambda|$. But $\dot{E}'$ is directly proportional to the magnitude of the vector in the ω_1,ω_2 space, since this quantity measures energy losses in a Coulomb damper, and $\dot{E}$ is proportional to the square of this vector magnitude, as appropriate for a viscous damper. Thus, inequality (1) predicts for this vector the existence of an asymptotic limit, giving Fig. 2 the appearance characteristic of a stable limit cycle. If the vehicle were subjected to a large initial perturbation, establishing the magnitude of the ω_1,ω_2 vector in excess of the limit cycle value (see point C), this vector magnitude would decay to the limit cycle value. Figure 3 provides an alternative interpretation of the same behavior illustrated in Fig. 2, portrayed in terms of the $\theta,\dot{\theta}$ phase space.

Figure 4 depicts the stability criterion directly: in the diagonally cross-hatched region the $\dot{E}'/\lambda'$ curve is above the $-\dot{E}/\lambda$ curve, so θ grows, approaching the limit cycle value. In the vertically cross-hatched region the situation is reversed, and θ diminishes toward the limit cycle value.

Figures 3 and 4 are conceptual representations of the dynamic response of dual-spin vehicles with small amounts of viscous damping on the platform and Coulomb damping (with stiction) on the rotor. For these figures, terms involving ζ and ζ' in Eq. (28) were ignored. However, when there is substantial damping or a critically tuned oscillator (r or r' nearly unity), the influences of ζ and ζ' may be significant. Exploring Eq. (28) in detail for the special case $n = 2, n' = 1$, one sees that there can be a stable limit cycle at one θ and an unstable limit cycle at some larger θ; i.e., the curves in Fig. 4 can cross *twice* when there is a significant linear (viscous) damper on the rotor. For this special case, substitution of the definitions for β, β', and Eq. (27) into Eq. (28) yields the stability criterion in the form

$$(2/\pi)(a\eta)^{-1}cG\beta^3\lambda^2 - \zeta'r'm'\beta'^2\lambda'^2 > 0 \tag{38}$$

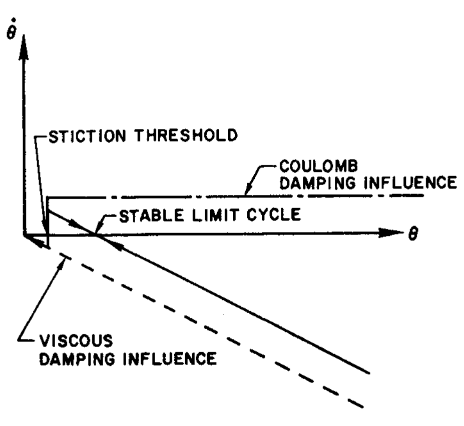

Fig. 3 Schematic for viscous platform damping and coulomb rotor damping.

In the algebraic manipulation between Eq. (28) and Eq. (38), use has been made of Eq. (29), for c', and both terms have been multiplied by $2\eta^2/\pi$. The value $\pi/4$ has been substituted for G', and the negativeness of λ' has been assumed [see Eq. (10)].

The critical values of θ corresponding to a neutral state (and here referred to as limit cycle values) are obtained by changing Eq. (38) to an equality and solving for $\tan\theta$ (implicit in β and β'). To meet this objective, we solve for the explicit value of β, using the correct expression for ζ for $n = 2$, from Eq. (32). We set $n = 2$ in Eq. (32), multiply by $(2r/\pi pm)^2$ and use the β definition squared to obtain

$$(4cG/a\eta\pi m)^2\beta^4 + (r^2 - 1)^2\beta^2 - [a\eta(\Omega - \lambda)\lambda^{-2}\tan\theta]^2 = 0 \tag{39}$$

If the constants a_1, a_2, a_3 are defined as necessary to rewrite Eq. (39) in the form

$$a_1\beta^4 + a_2\beta^2 - a_3\tan^2\theta = 0 \tag{40}$$

then the solution for β may be written

$$\beta = \{[-a_2 + (a_2^2 + 4a_1a_3\tan^2\theta)^{1/2}]/2a_1\}^{1/2} \tag{41}$$

with algebraic signs chosen to preserve the proper sign of β. This solution can finally be substituted into the stability criterion of Eq. (38), changed to an equality to provide the neutral or limit cycle values of θ, here designated θ^*. Substitution of Eq. (41) and β' into Eq. (38) yields

$$\frac{2cG\lambda^2}{\pi a\eta}\left\{\frac{1}{2a_1}[-a_2 + (a_2^2 + 4a_1a_3\tan^2\theta^*)^{1/2}]\right\}^{3/2} - \frac{\zeta'r'm'[a'\eta(\Omega + \sigma - \lambda')]^2\tan^2\theta^*}{\lambda'^2[(r'^2 - 1)^2 + 4\zeta'^2r'^2]} = 0 \tag{42}$$

This finally is an equation which one can attempt to solve for θ^*, the limit cycle values of θ. If it is multiplied by $(2a_1)^{3/2} \times \pi a\eta/2cG\lambda^2$, and the resulting coefficient of $\tan^2\theta^*$ in the second term is called a_4, then

$$[-a_2 + (a_2^2 + 4a_1a_3\tan^2\theta^*)^{1/2}]^{3/2} = a_4\tan^2\theta^* \tag{43}$$

Let us introduce the coordinate transformation

$$y^{3/2} \equiv \tan^2\theta^*, \tag{44}$$

take each side to the power $\frac{2}{3}$, add a_2 to each side, square the result, and factor to obtain,

$$y[4a_1a_3y^{1/2} - a_4^{4/3}y - 2a_2a_4^{2/3}] = 0 \tag{45}$$

The solution $y = 0$ (so $\theta^* = 0$) may now be noted, but attention focuses on the possibility that the expression in brackets may be zero for really, i.e., with implied definitions for a_5 and a_6

$$y^{1/2} = a_5y + a_6 \tag{46}$$

Squaring Eq. (46) and solving for y produces

$$y = -b_1 \pm (b_1^2 - b_2)^{1/2} \tag{47}$$

where $b_1 \equiv (2a_5a_6 - 1)/(2a_5^2)$, and $b_2 \equiv (a_6/a_5)^2$. Reversing the transformation $y^{3/2} = \tan^2\theta^*$, we obtain an explicit expression for limit cycle values of θ in the form

$$\tan^{4/3}\theta^* = -b_1 \pm (b_1^2 - b_2)^{1/2} \tag{48}$$

Since b_2 is positive, there will exist two real roots for $\tan\theta^*$

whenever

$$b_1^2 - b_2 > 0; \quad b_1 < 0 \tag{49}$$

Substitution of the foregoing definitions of b_1 and b_2 indicates that the single requirement for the existence of two limit cycle values for θ is

$$4a_5a_6 < 1 \tag{50}$$

Inequality (50) involves a complex combination of system parameters, but it is readily satisfied for configurations of interest.

A base line vehicle chosen for detailed study has the following fixed parameters: inertias A, B, and $J_3' = 410$, 200, and 136 slug-ft^2; $M = 17.3$ slug; $M' = 20$ slug; $m = m' = 0.31$ slug; $\Omega = 0$; $\sigma = 6$ rad/sec; $a = a' = 2$ ft; and $l = 5$ ft. The calculated nutation rates are $\lambda = 1.99$/sec and $\lambda' = -4.0$/sec. Quantities r, r', n, n', c and c' (or ζ and ζ') were varied to investigate dynamic response. The existence of two limit cycle values has been established from Eq. (48) for this vehicle with $n = 2$, $n' = 1$, $r = r' = 2$, $c = 246.79$ lb-sec/ft, and $c' = 0.263$ lb-sec/ft (corresponding to $\zeta' = 0.053$). Figure 5 indicates that the sum $\dot{E}/\lambda + \dot{E}'/\lambda'$ is alternately positive, negative, and positive as θ is increased. Stable and unstable limit cycles are predicted at $\theta^* = 0.0182$ rad and $\theta^* = 0.0885$ rad, respectively.

Figures 2–5 are all conceptual representations of dual-spin vehicle behavior, as indicated by the preceding heuristic analyses. When the objective is a qualitative appreciation of the character of the rotational motions of a dual-spin vehicle, such pictorial representations are of value. It should not be imagined however, that either these figures or the preceding energy-sink stability criteria are necessarily quantitatively valid indications of vehicle behavior. To obtain sound quantitative data one must confront directly the equations of motion of a specific system. Since nonlinearities are an important feature of the systems of interest, these equations can be solved, in general, only by digital computer numerical integration.

Results of Digital Simulations

The nonlinear equations of motion (2–6) for the specific dual-spin system of Fig. 1 have been programed for numerical integration, and investigations have been conducted with damping laws for D and D' as represented by Eq. (19). Thus, the program is written to accommodate on both rotor and platform either linear, sublinear, or superlinear damping, including Coulomb damping with simulated stiction. All computer runs were for the base line vehicle described in the text supporting Fig. 5. Duration of simulated vehicle response was typically limited to 60 sec. Of the many simulation series performed, three are most notable; they are described here as cases I, II, and III.

Case I

$n = 1$, $n' = 0$, $r = 1.5$, $r' = 2.0$, $c = 0.46273$ lb-sec/ft (so $\zeta = 0.25$), and $c' = 0.0095745$ lb-sec/ft. For this case, with viscous damper on the despun platform B and Coulomb damping on the rotor B', a limit cycle value $\theta^* = 0.0232$ rad is predicted by the combination of Eqs. (28), (29) and the one for ζ' corresponding to (32). Six different initial values θ_0 were adopted for θ, and the time changes in θ were noted, and sometimes plotted to assess stability. Results are as follows, for values of θ_0 in rad: 0.01, unstable; 0.017, unstable; 0.0232, very slightly unstable; 0.03, stable; 0.05, stable; 0.17, stable, with time constant 17 min. (The instability noted for $\theta_0 = 0.01$ had a negative time constant of similar magnitude.) Thus, when $\theta = 0.17$ rad $\cong 10°$, this vehicle would appear to be functioning properly, approaching the design state with $\theta = 0$ at such a rate that θ is diminished to $1/e$ of its value in 17 min. But θ would never go below the stable limit cycle at about 1.3°. If somehow the design null state were imposed on the vehicle, stiction might prevent the destabilizing damper on the rotor from becoming activated until some perturbation brought θ to some critical value (a small fraction of a degree), and then θ would increase to 1.3°. Figure 6 shows the time behavior of θ as indicated by numerical integrations from initial values $\theta_0 = 0.01$, 0.0232, and 0.17 rad.

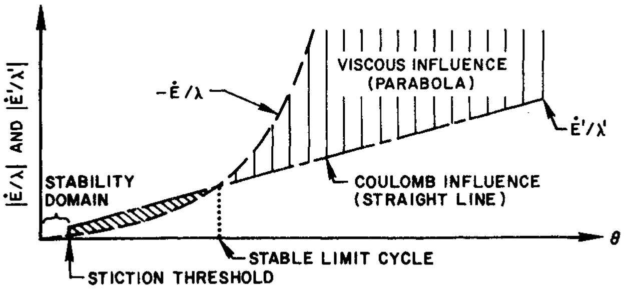

Fig. 4 Schematic showing stability criterion variation with coning angle (see Fig. 3).

Case II

This is the previously specified Fig. 5 case, for which numerical integrations commenced from four different initial conditions θ_0 (rad), with the following results: 0.0017, unstable; 0.017, slightly unstable; 0.034 stable; and 0.17, unstable. These results are entirely in harmony with the energy sink predictions.

The computer program also printed out average values of $\dot{E}/\lambda$ and $\dot{E}'/\lambda'$ over each nutation cycle. For most of the runs, and all of those reported as cases I and II, this print-out conformed to that predicted manually, and correlated well with stability as judged by the time behavior of θ. (Computer data were in fact used in constructing Fig. 5.) This correlation was however not always in evidence, as indicated in case III.

Case III

$n = 2$, $n' = 1$, $r = 1.1$, $r' = 2.0$, $c = 0.67867$ lb-sec/ft, $c' = 3.0773$ lb-sec/ft (so $\zeta' = 0.62$). From Eq. (48) it was predicted that a stable limit cycle should occur at $\theta^* = 0.012$ rad, and numerical integrations were performed for $\theta_0 = 0.0017$, 0.012, 0.017, and 0.17 rad. The first and third of these runs exhibited an unexpected oscillation in θ of about 27-sec period, which created a situation in which θ was sometimes diminishing when the sum $\dot{E}/\lambda + \dot{E}'/\lambda'$ was positive

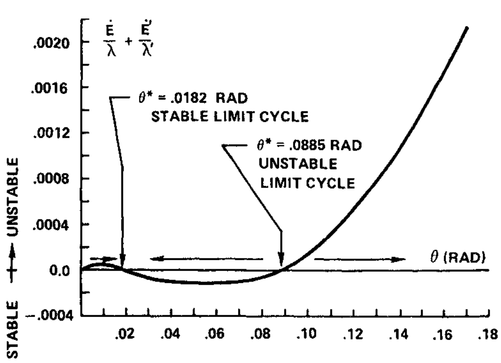

Fig. 5 Plot for base-line vehicle with viscous rotor damping and quadratic platform damping.

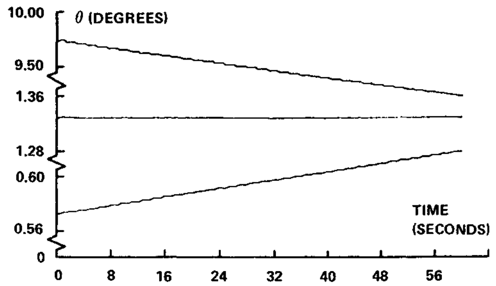

Fig. 6 Numerical integration for base-line vehicle with coulomb rotor damping and viscous platform damping.

according both to the manual prediction and the computer printout. In the secular sense (with the 27-sec periodic motion removed), θ behaved qualitatively as predicted before integration. Thus, even in this case the energy sink method gave meaningful results, although the method failed to predict the short-term behavior of θ. It would appear that the sharply tuned platform damper ($r = 1.1$) resulted in a beating phenomenon, with mechanical energy flowing back and forth between the oscillator motion and the gross vehicle motion.

Conclusions

It has been demonstrated unequivocally that an idealized torque-free dual-spin vehicle of the type illustrated in Fig. 1 can with the proper combination of generally nonlinear damping laws characterizing its oscillators exhibit the kind of limit cycle behavior illustrated in Figs. 2–5. The excellent relationship between the predictions of the energy-sink stability criterion and the results of numerical integration has been illustrated by example for this class of vehicle.

Much more subject to discussion and interpretation is the suggestion that the nonlinear damping limit cycle behavior of the idealized vehicle of Fig. 1 is relevant to the explanation of the flight anomalies of Tacsat 1, which maintained the pointing accuracy of its despun antenna within mission specifications but exhibited on occasion an anomalous coning motion θ remaining near 1°.[14] This innocuous but puzzling behavior has been explained[14] in terms of the dissipation characteristics of the bearing assembly between the rotor and the despun platform, on the basis of post-flight dynamic tests on a similar assembly. Although test data do not permit a simple characterization of the damping properties of the bearing as "Coulomb damping" or any such simple functional description, the presence of nonlinearities (and probably time and temperature dependencies) in that damping law is clear. Thus it seems correct to claim that the Tacsat 1 flight behavior is in a general sense "explained" by the mechanism described in idealized terms here.

The importance of a general interpretation of Tacsat 1 behavior in analytically tractable terms is clear to those responsible for future dual-spin spacecraft. The impossibility of designing a viscous damper for the despun platform which can completely cancel the destabilizing tendencies of any sublinear damping in the rotor is particularly noteworthy; one must either eliminate the latter, modify the former to introduce appropriate nonlinearities into the damping law, or tolerate limit cycle operation. In the case of Coulomb damping on the rotor, stiction may preserve a small domain of stability.

There may exist stable limit cycles attributable to nonlinearities other than those postulated here for the damping law; these may result, for example, from the presence of nonlinear springs (as we will report in a subsequent paper) or even from inherent nonlinearities in the system equation that are present with linear springs and dashpots and become important for large θ.[15]

The dual-spin attitude stabilization concept continues to offer promise, but it is now clear that its implementation requires careful preflight determination of mechanical properties and intensive dynamic analysis of quite complex mathematical models of the spacecraft.

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